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LETTER TO THE EDITOR

The q-deformed differential operator algebra, a new solution to the Yang–Baxter equation and quantum plane

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Abstract. The q-deformed differential calculus is proposed and analysed in the framework of quantum plane. The q-deformed differential operator algebra is investigated and applied in the investigation of the quantum group $SU_q(2)$ and its representations. The generalization to n-dimensional differential calculus is made and shown to be a new solution to quantum plane by providing a new solution of the Yang–Baxter equation.

In this letter we propose a q-deformed differential operator algebra (q-DDOA) and its realization of the quantum groups. These differential operators are most properly understood in the framework of a new non-commutative geometric calculus (quantum plane). The exchange matrix of the multi-dimensional q-DDOA is shown to be a new solution to the Yang–Baxter equation. The connection between the present quantum plane and that firstly proposed by Woronowicz [1] and further discussed by Wess and Zumino [2] is explored.

Let us recall the ordinary differential operator algebra (DOA) spanned by x, ∂, $x\partial$, with ($\partial = \partial/\partial x$) satisfying the following relations,

$$[x, \partial] = -1 \qquad [x\partial, x] = x \qquad [x\partial, \partial] = -\partial. \tag{1}$$

The representation is the Bargmann space

$$B = \left\{ f(n) = \frac{x^n}{\sqrt{n!}},\ n \in \mathscr{Z}^+,\ x \in \mathscr{R}^1 \right\}. \tag{2}$$

The operators act in B to yield

$$\begin{aligned} xf(n) &= \sqrt{n+1}\, f(n+1) \\ \partial f(n) &= \sqrt{n}\, f(n-1) \\ x\partial f(n) &= nf(n). \end{aligned} \tag{3}$$

It is well known that the semisimple Lie algebra can be realized in differential operators. Suppose that $M_{n\times n}$ is representation of semisimple Lie algebra g, then $(x_1, x_2, \ldots, x_n)^T M_{n\times n}(\partial_1, \partial_2, \ldots, \partial_n)$ is the differential realization of g.

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Now we investigate a q-analogous realization for quantum algebras by introducing an additional derivative operator, the q-derivative $\tilde{\partial}$ defined via its action on Bargmann states x^n,

$$\tilde{\partial} x^n = \{n\}_q x^{n-1} \qquad \{x\}_q = \frac{1-q^{2x}}{1-q^2}. \tag{4}$$

It is easy to find that the commutation relation between the variable and the derivative operator is twisted, namely,

$$\tilde{\partial} x = 1 + q^2 x \tilde{\partial}. \tag{5}$$

When $q \to 1$, the untwisted commutation relation is recovered, i.e.

$$\partial x = x\partial + 1. \tag{6}$$

By analogy with the ordinary commutation relations of the operators x, ∂, $x\partial$,

$$[x, \partial] = -1 \qquad [x\partial, x] = x \qquad [x\partial, \partial] = -\partial \tag{7}$$

we have the twisted relations,

$$[x, \tilde{\partial}] = -q^{2x\partial} \qquad [x\partial, x] = x \qquad [x\partial, \tilde{\partial}] = -\tilde{\partial}. \tag{8}$$

The algebraic relations (7) are usually denoted $D(1)$, i.e. the one-dimensional differential operator algebra, while the relations (8) are called the (one-dimensional) q-deformed differential operator algebra, namely, $D_q(1)$.

The studies on the representation of the algebra $D_q(1)$ are of two parts with respect to q being a root of unity or not. For q not a root of unity, the Bargmann representation for the q-DDOA is

$$\Gamma_\infty = \left\{ f(n) = \frac{x^n}{\sqrt{\{n\}_q!}},\ n \in \mathscr{Z}^+ \right\} \tag{9}$$

which is irreducible and indecomposable. When the operators act in this space,

$$\begin{aligned} xf(n) &= \sqrt{\{n+1\}_q} f(n+1) \\ \tilde{\partial} f(n) &= \sqrt{\{n\}_q} f(n-1) \\ x\tilde{\partial} f(n) &= \{n\}_q f(n) \\ x\partial f(n) &= n f(n). \end{aligned} \tag{10}$$

When q is a root of unity, i.e. $q^p = \pm 1$, where p is the possibly smallest positive integer. Therefore when $\{p\}_q = 0$, the representations (9) may be ill-defined. But it is easily seen that the states in the following representation are still well defined:

$$\Gamma'_\infty = \left\{ f'(n) = \frac{x^n}{\{n\}_q!},\ n \in \mathscr{Z}^+ \right\}. \tag{11}$$

The actions of the operators in this space yield

$$\begin{aligned} xf(n) &= \{n+1\}_q f(n+1) \\ \tilde{\partial} f(n) &= f(n-1) \\ x\tilde{\partial} f(n) &= \{n\}_q f(n) \\ x\partial f(n) &= n f(n). \end{aligned} \tag{12}$$

This representation is infinite dimensional, and every $|np-1\rangle$ state cannot be raised to $|np\rangle$ by the action of x. Therefore there are infinite number of invariant subspaces, denoted ρ_n. Each ρ_n consists of the states $|0\rangle, |1\rangle, \ldots, |np-1\rangle$, and

$$\rho_1 \subset \rho_2 \subset \rho_3 \subset \ldots \rho_n \subset \ldots \Gamma_\infty. \tag{13}$$

But state $|n\rangle$ can be raised to state $|n+p\rangle$ through the action of the Luzstig operator $L = x^p/\{p\}_q!$ [3].

Generalizing to $D_q(n)$, the n-dimensional differential operator algebra is straightforward, i.e.

$$\begin{aligned} &[x_i, \tilde{\partial}_j] = -\delta_{ij} q^{2x_i\partial_j} \\ &[x_i\partial_i, x_j] = \delta_{ij} x_i \qquad i, j = 1, 2, \ldots, n \\ &[x_i\partial_i, \tilde{\partial}_j] = -\delta_{ij}\tilde{\partial}_i \end{aligned} \tag{14}$$

and the basic relations between $\tilde{\partial}_i$ and x_j are

$$\tilde{\partial}_i x_j = \delta_{ij} + q^{2\delta_{ij}} x_j \tilde{\partial}_i. \tag{15}$$

The q-deformed Lie algebras or quantum groups such as those of A_n or C_n types can be realized via the above n-dimensional q-deformed differential operator algebra. The essential characteristics can be found in the specific example of $\mathrm{SU}_q(2)$ algebra, which is spanned by the generators X^+, X^- and X^0,

$$\begin{aligned} X^+ &= q^{-x_2\partial_2} x_1 \tilde{\partial}_2 \\ X^- &= q^{-x_1\partial_1} x_2 \tilde{\partial}_1 \\ X^0 &= \tfrac{1}{2}(x_1\partial_1 - x_2\partial_2) \end{aligned} \tag{16}$$

and it is a simple calculation to show the algebraic relations for $\mathrm{SU}_q(2)$

$$[X^+, X^-] = [2X^0]_q \qquad [X^0, X^\pm] = \pm X^\pm \tag{17}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1}) = \sinh \gamma x / \sinh \gamma$, $\gamma = \ln q$.

We consider the representations for $\mathrm{SU}_q(2)$ algebra. Let us start by looking into the case of q not a root of unity. The representations are

$$\Gamma_j = \left\{ |j, m\rangle = \frac{x_1^{j+m} x_2^{j-m}}{\sqrt{[j+m]_q! [j-m]_q!}}, \ m = -j, -j+1, \ldots, j \right\}. \tag{18}$$

When the operators act in this space, we get

$$\begin{aligned} (X^\pm)^a |j, m\rangle &= \sqrt{\frac{[j \mp m]_q! [j \pm m + a]_q!}{[j \mp m - a]_q! [j \pm m]_q!}} |j, m \pm a\rangle \\ X^0 |j, m\rangle &= m |j, m\rangle \\ C|j, m\rangle &= [j][j+1]|j, m\rangle \end{aligned} \tag{19}$$

where C is the Casimir operator for the algebra. The spaces Γ_j are of dimensions $D(j) = 2j+1$ and q-dimensions $D_q(j) = [2j+1]_q$, respectively. According to [4], all finite-dimensional representations of the quantum enveloping algebra are completely reducible and the irreducible ones can be classified in terms of highest weights.

If $q^p = \pm 1$, the representations for $\mathrm{SU}_q(2)$ proposed from those of the q-DDOA are

$$\Gamma'_j = \left\{ |j, m\rangle = \frac{x_1^{j+m} x_2^{j-m}}{[j+m]_q! [j-m]_q!}, \ m = -j, -j+1, \ldots, j-1, j \right\}. \tag{20}$$

And the actions of the generators in these spaces yield

$$\begin{aligned} X^-|j,m\rangle &= [j-m+1]_q|j,m-1\rangle \\ X^+|j,m\rangle &= [j+m+1]_q|j,m\rangle \\ \frac{(X^\pm)^a}{[a]_q}|j,m\rangle &= \frac{[j\pm m+a]_q!}{[a]_q![j+m]_q!}|j,m\pm a\rangle. \end{aligned} \tag{21}$$

It is also clear that $(X^\pm)^p=0$. And this property is preserved by the co-product $\Delta(X^\pm)^p=0$. All the finite-dimensional representations split into type one and type two. Those of type two are still like generic ones with spins $0\leqslant j\leqslant(n-2)/2$. Those of type one, however, are either irreducible and indecomposable and made of mixtures $(\rho_j,\rho_{j'})$ with $j'=-j-1 \bmod p$ and $|j-Bj'|<p$, or irreducible (like $\rho_{(p-1)/2}$). In any case they are characterized by a zero q-dimension.

The extension to the $A_q(n)$ algebra is straightforward. Let the elements be $\{X_i^+, X_i^-, X_i^0\}$, therefore

$$\begin{aligned} X_i^+ &= q^{-x_{i+1}}x_i\tilde{\partial}_{i+1} \\ X_i^- &= q^{-x_i}x_{i+1}\tilde{\partial}_i \qquad 1\leqslant i\leqslant n-1 \\ X_i^0 &= \tfrac{1}{2}(x_i\partial_i - x_{i+1}\partial_{i+1}). \end{aligned} \tag{22}$$

The basic properties of the representations can also be carried out by direct calculations in the Bargmann space. The possible extension to $C_q(n)$ algebra and other quantum algebras will be explored elsewhere.

It should be pointed out that there is Hopf algebraic structure hidden in q-DDOA. Let $L_+=x$, $L_-=q^{x\partial}\tilde{\partial}$, $L_0=x\partial$ then they span a Hopf algebra with the Hopf operations, the coproduct Δ, antipode S and co-unit ε defined in the following way

$$\begin{aligned} \Delta(L_0) &= L_0\otimes 1 + 1\otimes L_0 - \frac{\alpha}{\bar{\gamma}} \\ \Delta(L_+) &= (L_+\otimes q^{L_0/2} + \mathrm{i}q^{-L_0/2}\otimes L_+)\,\mathrm{e}^{-\mathrm{i}\alpha/2} \\ \Delta(L_-) &= (L_-\otimes q^{L_0/2} + \mathrm{i}q^{-L_0/2}\otimes L_-)\,\mathrm{e}^{-\mathrm{i}\alpha/2} \\ S(L_0) &= -L_0 + \mathrm{i}\frac{2\alpha}{\gamma}\cdot 1 \\ S(L_+) &= -q^{1/2}L_+ \\ S(L_-) &= -q^{1/2}L_- \\ \varepsilon(L_0) &= \frac{\alpha}{\bar{\gamma}} \\ \varepsilon(L_+) &= \varepsilon(L_-) = 0 \\ \varepsilon(1) &= 1 \end{aligned} \tag{23}$$

where $\alpha=2k\pi+\pi/2$, $k\in\mathscr{Z}$, $\bar{\gamma}=-\mathrm{i}\gamma$. Apparently, this Hopf algebra is isomorphic to the q-deformed oscillator algebra $H_q(4)$ [5]. When $q\to 1$, the ordinary differential operator algebra is recovered.

The existence of the Yang–Baxter equation (YBE) is a basic characteristic of quantum groups. From the q-deformed differential operator algebra, we can also construct the YBE. Firstly, the $\mathscr{R}$-matrix can be written explicitly in the following way:

$$\mathscr{R} = q^{1/2L_0 \otimes L_0 - (\alpha/\bar{\gamma})\Delta(L_0)} \sum_{n \geqslant 0} \mathrm{i}^n \frac{(1+q^{-1})^n}{[n]_{q^{1/2}}!} q^{-n(n+1)/4} (L_+)^n \otimes q^{-nL_0/2} L_-^n \tag{24}$$

where the convention

$$[x]_{(+,q)} = \frac{q^x + q^{-x}}{q + q^{-1}} = \frac{\cosh(\gamma x)}{\cosh \gamma} \tag{25}$$

is applied. The $\mathscr{R}$-matrix has the following properties which can all be verified by direct calculations

$$\begin{aligned} (\Delta \otimes \mathrm{id})\mathscr{R} &= \mathscr{R}_{13}\mathscr{R}_{23} \\ (\mathrm{id} \otimes \Delta)\mathscr{R} &= \mathscr{R}_{13}\mathscr{R}_{12} \\ (S \otimes \mathrm{id})\mathscr{R} &= \mathscr{R}^{-1} \end{aligned} \tag{26}$$

where the $\mathscr{R}_{ij}$ are the embeddings of $\mathscr{R}$ into $\mathscr{H}_q(1)^{\otimes 3}$. Hence we can show

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12} \tag{27}$$

which is just the Yang–Baxter equation.

Now we are in the position to show how the q-DDOA is most properly understood in the framework of non-commutative covariant calculus. As is pointed out in the above, the differential operator algebra defined by the basic commutation relations (6), (13) are modified by the deformation parameter q and share essential properties of the non-commutative geometry [6]. Though the basic variables x_i are commutative with each other, differently from Manin's proposal [6] of the non-commutative geometry and the covariant differential calculus defined for the construction of quantum plane by Wess and Zumino in [2], the new differential operator algebra supplies a new solution to the quantum plane proposal in [2], and consequently, a new solution to the Yang–Baxter equation.

According to [2], the basic variables, x_i, their differentials ξ_i and derivatives $\tilde{\partial}_i$ have the following relations

$$x_i x_j = B_{ij}^{kl} x_k x_l \qquad \xi_i \xi_j = -C_{ij}^{kl} \xi_k \xi_l \qquad \tilde{\partial}_i \tilde{\partial}_j = F_{ij}^{kl} \tilde{\partial}_k \tilde{\partial}_l. \tag{28}$$

where B, C and F are called exchange matrices. The consistent differential calculus may be defined if the required properties of the exterior differentials are satisfied:

$$\tilde{d} = \xi_i \tilde{\partial}_i \qquad \tilde{d}^2 = 0 \qquad \tilde{d}(fg) = (\tilde{d}f)g + f\tilde{d}g. \tag{29}$$

These lead to the following consistency constraints on the matrices B, C and F

$$\begin{aligned} &(E_{12} - B_{12})(E_{12} + C_{12}) = 0 \\ &(E_{12} + C_{12})(E_{12} - F_{12}) = 0 \\ &B_{12}C_{23}C_{12} = C_{23}C_2B_{23} \\ &C_{12}C_{23}F_{12} = F_{23}C_{12}C_{23} \\ &C_{12}C_{23}C_{12} = C_2C_{12}C_{23}. \end{aligned} \tag{30}$$

A non-trivial solution given in [2, solution (I)] is

$$B = F = \frac{1}{q}\hat{R} \qquad C = q\hat{R}$$

where $\hat{R}$ is the symmetrized R-matrix. This solution defines a consistent differential calculus on the quantum plane. It should be stressed that the $\hat{R}$ matrix in solution (I) has only two eigenvalues. And any $\hat{R}$ matrix with three or more eigenvalues is excluded by (I). The situation changes when we take the following non-trivial solution (II) to (28),

$$B = F = E \qquad C \propto \hat{R}. \tag{31}$$

In this case, the basic variables are commutative and the q-deformed commutation relations exist between differentials ξ_i and those between ξ and x_j, etc.

The commutation relations compatible with q-DDOA defined in (12) and (13) are

$$\begin{aligned}
&x_i x_j = x_j x_i \\
&\tilde{\partial}_i \tilde{\partial}_j = \tilde{\partial}_j \tilde{\partial}_i \\
&\xi_i \xi_j = -q^{2\delta_{ij}} \xi_j \xi_i \\
&\tilde{\partial}_i x_j = \delta_{ij} + q^{2\delta_{ij}} x_i \tilde{\partial}_j \\
&x_i \xi_j = q^{2\delta_{ij}} \xi_j x_i \\
&\tilde{\partial}_i \xi_j = q^{-2\delta_{ij}} \xi_i \tilde{\partial}_j
\end{aligned} \tag{32}$$

where $i, j = 1, 2, \ldots, n$. In this case the exchange matrix C is

$$C_{ij}^{kl} = q^{2\delta_{ij}} \delta_i^l \delta_j^k. \tag{33}$$

It can be easily checked that the above matrix satisfies (30). To the knowledge of the authors, it is a new solution to the Yang–Baxter equation. A further publication [7] will provide the relevant Yang–Baxterization, the construction of Wenzl algebra, especially the applications of this new solution in the physics theories.

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